University of California, Berkeley
Physics 105 Fall 2000 Section 1 (*Strovink*)

SOLUTION TO PROBLEM SET 8

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Reading:
105 Notes 9.1-9.8
Hand & Finch 7.1-7.10, 8.1-8.3

1.
Discuss the implications of Liouville's theorem on the focussing of beams of charged particles by considering the following simple case. An electron beam of circular cross section (radius R_0) is directed along the z axis. The density of electrons along the beam is constant, and the electrons all have the same z momenta, but their much smaller momentum components transverse to the beam (p_x or p_y) are distributed uniformly over a circle of radius p_0 in momentum space. If some focussing system is used to reduce the beam radius from R_0 to R_1, find the resulting distribution of the transverse momentum components. What is the physical meaning of this result? (Consider the angular divergence of the beam.)

Solution:
Since all the particles in the beam have the same z momenta, only four of the six dimensions of phase space are of interest: x, y, p_x, and p_y. The particles all lie in a circle in the x-y plane of radius R_0, and in a circle in the p_x-p_y plane of radius p_0. So the total volume in the four-dimensional phase space is the product of the areas of the circles: $V = \pi^2 R_0^2 p_0^2$. If we shrink the x-y circle from R_0 to R_1, we have to increase the radius of the momentum circle by the same factor to keep V constant. So the beam will now fill a circle in momentum space of radius $p_1 = p_0 \frac{R_0}{R_1}$. The ratio p_0/p_z is a measure of the angular divergence of the beam, so this result says that if you make the physical size of the beam smaller, you force it to become less well collimated, i.e. more divergent.

2.
In New Orleans (30° N latitude), there is a hockey arena with frictionless ice. The ice was formed by flooding a rink with water and allowing it to freeze slowly. This implies that a plumb bob would always hang in a direction perpendicular to the small patch of ice directly beneath it.

Show that a hockey puck (shot slowly enough that it stays in the rink!) will travel in a *circle*, making one revolution every day.

Solution:
The significance of the fact that the ice is perpendicular to a plumb bob is that there is no component of the combined (gravitational + centrifugal) force that is not cancelled by the normal force from the ice. (See the solution to Problem 4 for additional discussion of this point.)

Take ($\hat{z}$ = up [opposite to plumb bob direction], $\hat{y}$ = north, $\hat{x}$ = east) immediately above a patch of ice at latitude 30° (colatitude $\lambda \equiv 60°$). In this system, the angular velocity of the earth's rotation is directed along

$$\hat{\omega}_e = \hat{z}\cos\lambda + \hat{y}\sin\lambda \ .$$

If its instantaneous (horizontal) velocity is $\vec{v}$, with components v_x and v_y, a puck of mass m feels a Coriolis force

$$\vec{F}_{\rm Cor} = -2m\vec{\omega}_e \times \vec{v} \ .$$

The horizontal component of the force is

$$\begin{aligned} \vec{F}^{xy}_{\rm Cor} &= 2m(\hat{y}\,\omega_{ez}v_x - \hat{x}\,\omega_{ez}v_y) \\ &\equiv m\vec{\Omega}\times\vec{v} \quad \text{where} \\ \vec{\Omega} &\equiv -2\hat{z}\,\omega_{ez} \\ &= -2\hat{z}\,\Omega_e\cos\lambda \ . \end{aligned}$$

The effect of the horizontal component of the Coriolis force on the puck is

$$\begin{aligned} m\dot{\vec{v}} &= \vec{F}^{xy}_{\rm Cor} \\ &= m\vec{\Omega}\times\vec{v} \ . \end{aligned}$$

When the rate of change of a vector is always perpendicular to its instantaneous direction, the vector's length stays fixed. The last equation requires its direction to precess about $\vec{\Omega}$ with an angular velocity equal to the magnitude of Ω. Since $\vec{\Omega}$ is directed along $-\hat{z}$, the precession is clockwise. The period is

$$\frac{2\pi}{\Omega} = \frac{2\pi}{2\Omega_e \cos\lambda} = \frac{2\pi}{\Omega_e} = 1 \text{ day}.$$

3.
Consider a situation exactly the same as in the previous problem, except that the rink is centered at the *north pole*. This stimulates a controversy:

Simplicio: "The angular frequency of circular motion of the puck is $2\Omega_e \cos\lambda$ with $\cos\lambda = 1$ rather than $\frac{1}{2}$ as in the previous problem [where Ω_e is the angular velocity of the earth's rotation about its axis]. So $\omega_{\text{puck}} = 2\Omega_e$."

Salviati: "Work the problem in the [inertial] reference frame of the fixed stars. For a particular set of initial conditions, the puck can be motionless in this frame while the earth and rink rotate under it. Then $\omega_{\text{puck}} = \Omega_e$."

Who is right? Why?
Solution:
Simplicio is right. Even if the rink is at the north pole, the general method of analysis used in the previous problem is still valid.

What confuses Salviati? What he has in mind is a hypothetical case in which the puck remains at rest with respect to the fixed stars (it is OK to neglect the fact that the earth orbits the sun). The puck lies at a fixed high latitude, a short distance from the north pole, so that it remains on the ice of a rink that is centered at the pole. As the earth turns under the puck, the rink rotates under it as well. This occurs with a period of one day, not half a day as Simplicio calculates.

To prove Salviati wrong, we need to show that his hypothetical case can't occur. Once again, the key is the definition of "up", which is directly related to the shape of the ice. Salviati presumes that the puck can remain at rest with respect to the fixed stars. Therefore (in the frame of the fixed stars) there can be no net force on it. In order for this to happen, the earth's gravitational force must exactly balance the normal force from the ice. Therefore the surface of the ice must have a spherical shape, with a radius of curvature equal to the distance to the earth's center.

However, we are told that the ice is formed by flooding the rink and allowing the water to freeze. This requires the surface of a patch of ice to be normal to a plumb bob held over it. Because the plumb bob feels both the earth's gravitational force and a centrifugal force corresponding to its rotation, the plumb bob will point to a location south of the earth's center. This will cause the radius of curvature of the ice at the north pole rink to be larger than the earth's radius. Because the ice has this shallower curvature, a small component of the normal force on the puck will not be offset by gravity and will push the puck toward the pole. In order to stay at the same latitude, it would be necessary for the puck to travel in a circle (as viewed in the fixed stars!) so that it would have enough poleward acceleration to obey this poleward force. The combination of this absolute circular motion, and the rotation of the earth under it, produces a net rotation of the puck relative to the earth which is twice as fast as Salviati envisages.

4.
Consider a particle that is projected vertically upward from a point on the earth's surface at north latitude ψ_0 (measured from the equator). (Here "upward" means opposite to the direction that a plumb bob hangs.) Show that it strikes the ground at a point $\frac{4}{3}\omega\sqrt{(8h^3/g)}\cos\psi_0$ to the west, where ω is the earth's angular velocity and h is the height reached. [*Hints*: Neglect air resistance and consider only heights small enough that g remains constant. Simplify your algebra by using the fact that the Coriolis force is very small with respect to the gravitational force – more quantitatively $\omega T \ll 1$, where T is the flight's duration.]
Solution:
First, let's make sure that we understand why "vertically upward" is taken to be opposite to the

direction of a plumb bob. Let $\mathbf{g}_{\rm grav}$ be the gravitational acceleration vector. A mass hanging on a string will not align itself parallel to $\mathbf{g}_{\rm grav}$, because it will feel the centrifugal force as well as the gravitational force. The total (gravitational plus centrifugal) acceleration on a stationary object is $\vec{g} = \mathbf{g}_{\rm grav} - \vec{\omega} \times (\vec{\omega} \times \vec{r})$. So, if it were not already specified in the problem, we could imagine that the "vertical" direction could be chosen to be parallel either to $\mathbf{g}_{\rm grav}$ or to $\vec{g}$. If the latter is specified, than we can ignore centrifugal forces for the rest of the problem, since the gravitational plus centrifugal force has no component perpendicular to the "vertical" direction. (Note that the centrifugal force is of order 10^{-3} times the gravitational force, so the two directions differ by only a fraction of a degree, and $\vec{g}$ and $\mathbf{g}_{\rm grav}$ are nearly equal in magnitude.)

So we choose a coordinate system as follows: Let the z-direction be vertically up in the sense just discussed (including the centrifugal force along with the gravitational force). Then let the y-direction be west (tangent to the earth's surface), and the x-direction be perpendicular to both z and y. (So x points mostly north.) Then $\vec{g} = -g\hat{z}$, and the earth's angular velocity is $\vec{\omega} = \omega\,(\hat{x}\cos\psi_0 + \hat{z}\sin\psi_0)$. Our particle is subject to the acceleration $\vec{g}$ and the Coriolis acceleration:

$$\ddot{\vec{r}} = \vec{g} - 2\vec{\omega} \times \vec{v}$$

In cartesian components, these equations are

$$\begin{aligned} \ddot{x} &= 2\omega_z \dot{y} \\ \ddot{y} &= 2\omega_x \dot{z} - 2\omega_z \dot{x} \\ \ddot{z} &= -g - 2\omega_x \dot{y} \end{aligned}$$

and we have initial conditions $x = y = z = \dot{x} = \dot{y} = 0$, $\dot{z} = v_0$. The three differential equations can all be integrated once to yield

$$\begin{aligned} \dot{x} &= 2\omega_z y \\ \dot{y} &= 2\omega_x z - 2\omega_z x \\ \dot{z} &= v_0 - gt - 2\omega_x y \end{aligned}$$

It's not so easy to solve these equations exactly, so let's use the method of perturbations. (Perturbative methods look promising, since we know that the actual solution to the equations is going to be extremely close to the solution we get by neglecting the Coriolis force.) The Coriolis force is weak, which means that we can treat ω as a small quantity, and expand in powers of it. Let $x_0(t), y_0(t), z_0(t)$ be the solutions to these equations to zeroth order in ω. These quantities are just the solutions to the equations of a particle moving under the influence of ordinary gravity: With our initial conditions, the solutions are

$$x_0(t) = 0 \qquad y_0(t) = 0 \qquad z_0(t) = v_0 t - \tfrac{1}{2} g t^2$$

Our solution is a small perturbation of this, so we can set $x(t) = x_0(t) + x_1(t)$, along with similar expressions for y and z. Then x_1, y_1, and z_1 are all small. We shall retain only terms to first order in the small quantities ω, x_1, y_1, z_1. First we write the equation for $\dot{x}$:

$$\dot{x}_1 = 2\omega_z y_1 \ ,$$

where we have used the fact that $x_0 = y_0 = 0$. The right-hand side is second order in the small quantities. Neglecting it, we conclude that $x_1 \approx 0$. Therefore we're interested mainly in the westward displacement y_1. Let's write the equation for $\dot{y}$:

$$\dot{y}_1 = 2\omega_x z_0 - 2\omega_z x_0 = 2\omega_x (v_0 t - \tfrac{1}{2} g t^2)$$

Integrate this equation to get

$$y(t) = \omega_x \left(v_0 t^2 - \tfrac{1}{3} g t^3 \right)$$

From the z_0 equation, we find that the particle hits the ground at a time $T = 2v_0/g$, and the height is $h = v_0^2/2g$. So the westerly displacement is

$$y_1(T) = \frac{4}{3}\omega_x \left(\frac{8h^3}{g} \right)^{1/2} = \frac{4}{3}\omega \cos\psi_0 \left(\frac{8h^3}{g} \right)^{1/2}$$

5.

Consider the description of the motion of a particle in a coordinate system that is rotating with uniform angular velocity ω with respect to an inertial reference frame. Use cylindrical coordinates, taking $\hat{z}$ to lie along the axis of rotation, and assume that the ordinary potential energy U is velocity-independent. Obtain the Lagrangian for the particle in the rotating system. Calculate the Hamiltonian and identify this quantity with the total energy E. Show that $E = \frac{1}{2}mv^2 + U + U'$, where U is the ordinary potential energy and U' is a pseudopotential. How does U' depend on the cylindrical coordinate r?

Solution:

The velocity of a particle in a rotating coordinate system is $\vec{v} = \dot{\vec{r}} + \vec{\omega} \times \vec{r}$. Let's use cylindrical coordinates r, ϕ, z, with ω in the z-direction. Then $\vec{\omega} \times \vec{r} = \omega r \hat{\phi}$, so $\vec{v} = \dot{r}\hat{r} + \dot{z}\hat{z} + r(\dot{\phi} + \omega)\hat{\phi}$. The Lagrangian is therefore

$$\mathcal{L} = \tfrac{1}{2}mv^2 - U = \tfrac{1}{2}m\left(\dot{r}^2 + r^2(\dot{\phi}+\omega)^2 + \dot{z}^2\right) - U.$$

The canonical momenta are defined by $p_j = \partial\mathcal{L}/\partial\dot{q}_j$, so

$$p_r = m\dot{r} \qquad p_\phi = mr^2(\dot{\phi}+\omega) \qquad p_z = m\dot{z}$$

Use the definition of the Hamiltonian

$$\begin{aligned}\mathcal{H} &= p_j\dot{q}_j - \mathcal{L} \\ &= m\left(\dot{r}^2 + r^2(\dot{\phi}+\omega)\dot{\phi} + \dot{z}^2\right) - \mathcal{L} \\ &= \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - \tfrac{1}{2}mr^2\omega^2 + U\ .\end{aligned}$$

The first term is just the usual expression for kinetic energy, so it represents the "apparent kinetic energy" observed by someone in the rotating coordinate system. The second term is the effective potential, and the last term is the ordinary potential.

6.

Consider an Euler rotation

$$\begin{aligned}\tilde{x} &= \Lambda_\psi^t \tilde{x}''' \\ &= \Lambda_\psi^t \Lambda_\theta^t \tilde{x}'' \\ &= \Lambda_\psi^t \Lambda_\theta^t \Lambda_\phi^t \tilde{x}'\ ,\end{aligned}$$

where $\tilde{x}$ is a vector in the body axes and $\tilde{x}'$ is a vector in the space axes. Here

$$\Lambda_\phi^t \equiv \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda_\theta^t \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$\Lambda_\psi^t \equiv \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\ .$$

In the body axes, define

$$\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi\ ,$$

where

$$\begin{aligned}\vec{\omega}_\phi &\equiv \hat{x}_3'\dot{\phi} \\ \vec{\omega}_\theta &\equiv \hat{x}_1''\dot{\theta} \\ \vec{\omega}_\psi &\equiv \hat{x}_3'''\dot{\psi}\ .\end{aligned}$$

Find the components of $\vec{\omega}$ along the x_1', x_2', and x_3' (fixed) axes.

Solution:

At the outset, let's introduce some notation. Define the vectors $\hat{p} = \hat{x}_3'$, $\hat{q} = \hat{x}_1''$, and $\hat{r} = \hat{x}_3'''$. Then

$$\vec{\omega} = \dot{\phi}\hat{p} + \dot{\theta}\hat{q} + \dot{\psi}\hat{r}.$$

As a warmup, we write the components of each of the vectors $\hat{p}, \hat{q}, \hat{r}$ in the *unprimed* (body) coordinate system. Expressed in the primed (space) system, $\hat{p}'$ is $(0,0,1)$, so in the body system

$$\hat{p} = \Lambda_\psi^t \Lambda_\theta^t \Lambda_\phi^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta\sin\psi \\ \sin\theta\cos\psi \\ \cos\theta \end{pmatrix}$$

(We have used the explicit forms of the Λ-matrices.) Similarly,

$$\hat{q} = \Lambda_\psi^t \Lambda_\theta^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{pmatrix}$$

$$\hat{r} = \Lambda_\psi^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\ .$$

Substituting these into our expression for $\vec{\omega}$, we get

$$\vec{\omega} = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}$$

in the *unprimed* basis. So far this parallels Lecture Notes section 9.3.

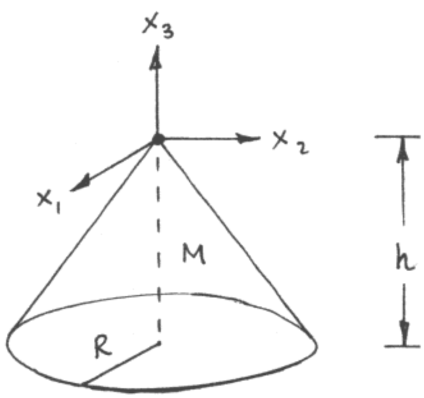
What we really need to do is to write $\hat{p}, \hat{q}, \hat{r}$ in the *primed* (fixed) basis. The first one is easy: $\hat{p}' = (0, 0, 1)$. Using the fact that the transformation matrices are orthogonal ($\Lambda_i^{-1} = \Lambda_i^t$), we get

$$\hat{q}' = \Lambda_\phi \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$$
$$\hat{r}' = \Lambda_\phi \Lambda_\theta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\phi\sin\theta \\ -\sin\theta\cos\phi \\ \cos\theta \end{pmatrix} .$$

So, in the *primed* basis, the components of $\vec{\omega}'$ are

$$\vec{\omega}' = \begin{pmatrix} \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\sin\theta \\ \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \\ \dot{\phi} + \dot{\psi}\cos\theta \end{pmatrix}$$

7.
Calculate the inertia tensor of a uniform right circular cone of mass M, radius R, and height h. Take the x_3 direction to be along the cone's axis. For this calculation, take the origin to be...



(a)
... at the apex of the cone (as shown in the figure).
Solution:
Remember the formula

$$\mathcal{I}_{ij} = \int \rho(\vec{x}) \left(\delta_{ij}\vec{x}^2 - x_i x_j\right) d^3x$$

where $\rho(\vec{x})$ is the density at point $\vec{x}$. (In our case, ρ is just a constant.) All of the off-diagonal elements in our case are zero. To see why, consider $\mathcal{I}_{13}$:

$$\mathcal{I}_{13} = -\rho \int xz\, dx\, dy\, dz$$

But the region of integration is symmetric under the substitution $x \to -x$ (reflection about the yz plane), while the integrand changes sign under this reflection. So $\mathcal{I}_{13} = -\mathcal{I}_{13}$, which can only happen if $\mathcal{I}_{13} = 0$. All the off-diagonal elements of the inertia tensor vanish in the same way. (If you don't like this argument, you can just do the integrals explicitly.)

Now we need to compute the three diagonal elements. Let's start with $\mathcal{I}_{33}$. In cylindrical coordinates r, ϕ, z, for convenience directing z along $-x_3$, the integral is

$$\begin{aligned}
\mathcal{I}_{33} &= \rho \int \left(\vec{x}^2 - z^2\right) r\, dr\, d\phi\, dz \\
&= \rho \int_0^h dz \int_0^{Rz/h} dr \int_0^{2\pi} d\phi\, r^3 \\
&= 2\pi\rho \int_0^h dz \tfrac{1}{4} \left(\frac{Rz}{h}\right)^4 \\
&= \frac{\pi}{10} \rho R^4 h
\end{aligned}$$

The volume of a cone is $\frac{\pi}{3} R^2 h$, so $\rho = 3M/\pi R^2 h$. Thus

$$\mathcal{I}_{33} = \tfrac{3}{10} M R^2$$

Now let's find $\mathcal{I}_{11}$:

$$\begin{aligned}
\mathcal{I}_{11} &= \rho \int \left(y^2 + z^2\right) dx\, dy\, dz \\
&= \rho \int \left(r^2 \sin^2\phi + z^2\right) r\, dr\, d\phi\, dz \\
&= \rho \int_0^h dz \int_0^{Rz/h} dr \int_0^{2\pi} d\phi\, r \left(r^2 \sin^2\phi + z^2\right) \\
&= \pi\rho \int_0^h dz \int_0^{Rz/h} dr (r^3 + 2rz^2) \\
&= \frac{\pi\rho}{20} \left(R^4 h + 4R^2 h^3\right) \\
&= \tfrac{3}{20} M \left(R^2 + 4h^2\right) .
\end{aligned}$$

Of course, $\mathcal{I}_{11} = \mathcal{I}_{22}$ by symmetry.

(**b**)

... at the cone's center of mass.

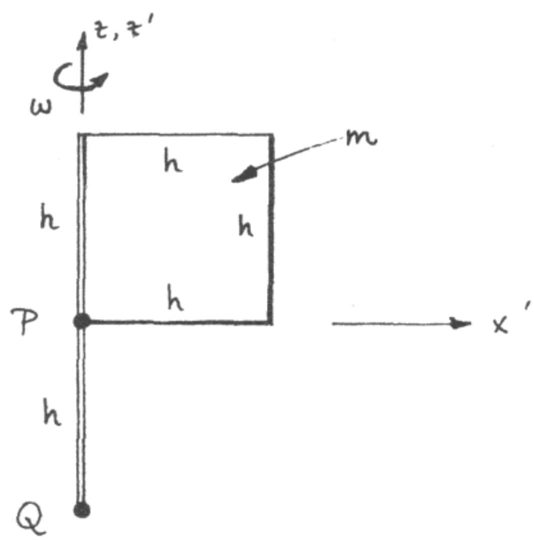
Solution:

The parallel axis theorem says that moments of inertia about an axis that is displaced by a distance d from one of the principal axes passing through the center of mass is greater than the corresponding moment about the center of mass by the amount Md^2. In our case, the center of mass is a distance $d = \frac{3}{4}h$ in the z-direction away from the apex. So $\mathcal{I}_{33}$ is unaffected by moving to the center of mass, while $\mathcal{I}_{11}$ and $\mathcal{I}_{22}$ decrease by Md^2. That is,

$$\begin{aligned}\mathcal{I}_{11} = \mathcal{I}_{22} &= \tfrac{3}{20}M\left(R^2 + 4h^2\right) - \tfrac{9}{16}Mh^2 \\ &= \tfrac{3}{20}M\left(R^2 + \tfrac{1}{4}h^2\right) \\ \mathcal{I}_{33} &= \tfrac{3}{10}MR^2 \ .\end{aligned}$$

8.

A square door of side h and mass m rotates with angular velocity ω about the z' (space) axis. The door is supported by a stiff light rod of length $2h$ which passes through bearings at points P and Q. P is at the origin of the primed (fixed) and unprimed (body) coordinates, which are coincident at $t = 0$. Neglect gravity.



(**a**)

Calculate the angular momentum **L** about P in the body system.

Solution:

In the body axes, let's find the elements of the inertia tensor about the point P. The surface density is $\rho = m/h^2$, and all of our integrals need only be taken over x and z, with $y = 0$.

$$\begin{aligned}\mathcal{I}_{11} &= \rho\int_0^h\int_0^h z^2\,dx\,dz = \tfrac{1}{3}mh^2 \\ \mathcal{I}_{33} &= \rho\int_0^h\int_0^h x^2\,dx\,dz = \tfrac{1}{3}mh^2 \\ \mathcal{I}_{22} &= \rho\int_0^h\int_0^h (x^2+z^2)\,dx\,dz = \tfrac{2}{3}mh^2 \\ \mathcal{I}_{12} &= -\rho\int_0^h\int_0^h xy\,dx\,dz = 0 \\ \mathcal{I}_{23} &= -\rho\int_0^h\int_0^h yz\,dx\,dz = 0 \\ \mathcal{I}_{13} &= -\rho\int_0^h\int_0^h xz\,dx\,dz = -\tfrac{1}{4}mh^2 \ .\end{aligned}$$

Using the fact that $\vec{\omega} = \omega\hat{z}$,

$$\begin{aligned}\vec{L} &= \mathcal{I}\vec{\omega} \\ &= mh^2\omega\begin{pmatrix}\frac{1}{3} & 0 & -\frac{1}{4} \\ 0 & \frac{2}{3} & 0 \\ -\frac{1}{4} & 0 & \frac{1}{3}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} \\ &= mh^2\omega\begin{pmatrix}-\frac{1}{4}\\0\\\frac{1}{3}\end{pmatrix} .\end{aligned}$$

So $\vec{L} = mh^2\omega(-\frac{1}{4}\hat{x} + \frac{1}{3}\hat{z})$.

(**b**)

Transform to get $\mathbf{L}'(t)$ in the fixed system.

Solution:

To convert to fixed (space) axes, note that $\hat{z} = \hat{z}'$, and $\hat{x} = \hat{x}'\cos\omega t + \hat{y}'\sin\omega t$. So

$$\vec{L}' = mh^2\omega\left(-\tfrac{1}{4}(\hat{x}'\cos\omega t + \hat{y}'\sin\omega t) + \tfrac{1}{3}\hat{z}'\right) .$$

(**c**)

Find the torque $\mathbf{N}'(t)$ exerted about the point P by the bearings.

Solution:

$$\vec{N}' = \dot{\vec{L}}' = \tfrac{1}{4}mh^2\omega^2(\hat{x}'\sin\omega t - \hat{y}'\cos\omega t) \ .$$

(**d**)

Assuming that the bearing at P exerts no torque about P, find the force $\mathbf{F}'_Q(t)$ exerted by the bearing at Q.

Solution:

Taking $\vec{r}'$ to be a vector from P to Q, the torque $\vec{N}' = \vec{r}' \times \vec{F}'$. So the force $\vec{F}'$ at point Q causes a torque

$$\vec{N}' = h(F'_{y'}\hat{x}' - F'_{x'}\hat{y}') \ .$$

Setting this equal to the torque from the previous part, we get

$$\vec{F}' = \tfrac{1}{4}mh\omega^2(\hat{x}'\cos\omega t + \hat{y}'\sin\omega t) \ .$$

(In the body axes, this force points in a constant ($\hat{x}$) direction.)